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ANNALS OF MATHEMATICS.

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LAGRANGE'S SEXTIC.

By PROF. ALBERT M. SAWIN, Laramie, Wyo.

It has been for a long time considered that Lagrange's method for the reduction of the quintic equation depended upon the solution of an auxiliary equation of the sixth degree, known and designated by him as a "Sextic Resolvent." While this equation has been the subject of a good deal of valuable literature, and, with its analogues, occupies an important place in the theory of equations, I am not aware that it has been shown that this resolvent of a higher degree is not necessary to the application of Lagrange's method. It is the object of the present paper to show that Lagrange's "Sextic Resolvent" is not necessary to the resolution of the quintic equation in the application of his own method for its reduction, and neither does it require a resolvent equation of a higher degree than the fifth degree for this purpose.

Denoting the five roots of the quintic by a, b, c, d, e, and the roots of $\omega^5 - 1 = 0$ by $1, \omega, \omega^2, \omega^3, \omega^4$, we may write the entire six Lagrangian groups as follows:

$$M \left| \begin{array}{l} (a + \omega b + \omega^2 c + \omega^3 d + \omega^4 e)^5 \\ (a + \omega^2 b + \omega^4 c + \omega d + \omega^3 e)^5 \\ (a + \omega^3 b + \omega c + \omega^4 d + \omega^2 e)^5 \\ (a + \omega^4 b + \omega^3 c + \omega^2 d + \omega e)^5 \end{array} \right.$$

$$N \left| \begin{array}{l} (a + \omega c + \omega^2 b + \omega^3 d + \omega^4 e)^5 \\ (a + \omega^2 c + \omega^4 b + \omega d + \omega^3 e)^5 \\ (a + \omega^3 c + \omega b + \omega^4 d + \omega^2 e)^5 \\ (a + \omega^4 c + \omega^3 b + \omega^2 d + \omega e)^5 \end{array} \right.$$

$$P_1 \begin{cases} (a+\omega d+\omega^2 c+\omega^3 b+\omega^4 e)^5 \\ (a+\omega^2 d+\omega^4 c+\omega b+\omega^3 e)^5 \\ (a+\omega^3 d+\omega c+\omega^4 b+\omega^2 e)^5 \\ (a+\omega^4 d+\omega^3 c+\omega^2 b+\omega e)^5 \end{cases}$$

$$Q_1 \begin{cases} (a+\omega e+\omega^2 c+\omega^3 d+\omega^4 b)^5 \\ (a+\omega^2 e+\omega^4 c+\omega d+\omega^3 b)^5 \\ (a+\omega^3 e+\omega c+\omega^4 d+\omega^2 b)^5 \\ (a+\omega^4 e+\omega^3 c+\omega^2 d+\omega b)^5 \end{cases}$$

$$R_1 \begin{cases} (a+\omega b+\omega^2 e+\omega^3 d+\omega^4 c)^5 \\ (a+\omega^2 b+\omega^4 e+\omega d+\omega^3 c)^5 \\ (a+\omega^3 b+\omega e+\omega^4 d+\omega^2 c)^5 \\ (a+\omega^4 b+\omega^3 e+\omega^2 d+\omega c)^5 \end{cases}$$

$$S_1 \begin{cases} (a+\omega b+\omega^2 c+\omega^3 e+\omega^4 d)^5 \\ (a+\omega^2 b+\omega^4 c+\omega e+\omega^3 d)^5 \\ (a+\omega^3 b+\omega c+\omega^4 e+\omega^2 d)^5 \\ (a+\omega^4 b+\omega^3 c+\omega^2 e+\omega d)^5 \end{cases}$$

Any one of the 24 values here written is a root of a quartic consequent upon the solution of a sextic; and if we consider the four imaginary values corresponding to the roots of unity, viz. ω, ω^2, ω^3, ω^4, as the four values of group M_1, any symmetrical function of these, or rather their fifth powers, as for instance their sum, is a six-valued function of the roots, and is therefore determined by a sextic equation.

If we separate these six groups into three pairs, viz.:

$$w_1 = M_1 + Q_1, \qquad w_2 = N_1 + S_1, \qquad w_3 = P_1 + R_1,$$

these pairs do not break up for any of the substitutions of the roots b, c, d, e; the pairs simply interchange places. For example, substitute in these six groups the roots b and c and we have, M_1 and N_1 change places, Q_1 and S_1 change places, P_1 and R_1 change places, or w_1 and w_2 change places and w_3 remains constant.

The substitution of the groups caused by all the substitutions of the roots b, c, d, e may be seen in the following table:

	bc	bd	be	cd	ce	de
$M_1 + Q_1$	$N_1 + S_1$	$P_1 + R_1$	$Q_1 + M_1$	$Q_1 + M_1$	$R_1 + P_1$	$S_1 + N_1$
$N_1 + S_1$	$M_1 + Q_1$	$S_1 + N_1$	$P_1 + R_1$	$P_1 + R_1$	$S_1 + N_1$	$Q_1 + M_1$
$P_1 + R_1$	$R_1 + P_1$	$M_1 + Q_1$	$N_1 + S_1$	$N_1 + S_1$	$Q_1 + M_1$	$R_1 + P_1$

from which it is evident that the pairs in all the columns remain unchanged for these substitutions; hence any symmetrical functions of these pairs or any symmetrical function of w_1, w_2, w_3 is a symmetrical function of b, c, d, e, and peradventure said function should also be a symmetrical function of $M_1, N_1, P_1, Q_1, R_1, S_1$, it would also be a symmetrical function of all five of the roots a, b, c, d, e.

Now, if w_1, w_2, w_3 be the roots of

$$W^3 - A_1 W^2 + B_1 W - C_1 = 0,$$

A_1 is evidently a symmetrical function of the roots of the quintic, and B_1 and C_1 are symmetrical functions of b, c, d, and e only, and therefore change for the substitutions ab, ac, ad, and ae, and are therefore five-valued functions of the roots, whose determination therefore requires the solution of a quintic equation, whence the roots w_1, w_2, w_3 become known.

Next let the quartic roots in group M_1 be taken as the sum of their products two and two at a time, and so on for all the other groups, thus making six other groups $M_2, N_2, P_2, Q_2, R_2, S_2$, the subscripts denoting the number of roots taken together in each product.

Denoting as before the similar pairs

$$x_1 = M_2 + Q_2, \qquad x_2 = N_2 + S_2, \qquad x_3 = P_2 + R_2,$$

we have $X^3 - A_2 X^2 + B_2 X - C_2 = 0$, whence by the aid of subordinate quintics as before the values x_1, x_2, x_3 become known. Similarly, treating the groups in sums of the products of the quartic roots taken three at a time and four at a time, we have

$y_1 = M_3 + Q_3$, $y_2 = N_3 + S_3$, $y_3 = P_3 + R_3$, whence y_1, y_2, y_3 become known;
$z_1 = M_4 + Q_4$, $z_2 = N_4 + S_4$, $z_3 = P_4 + R_4$, " z_1, z_2, z_3 " "

Instead of taking the above pairs as their sums, it is obvious that the above reasoning may be applied to their products as well; thus, letting

$$\begin{aligned} w'_1 &= M_1Q_1, & w'_2 &= N_1S_1, & w'_3 &= P_1R_1; \\ x'_1 &= M_2Q_2, & x'_2 &= N_2S_2, & x'_3 &= P_2R_2; \\ y'_1 &= M_3Q_3, & y'_2 &= N_3S_3, & y'_3 &= P_3R_3; \\ z'_1 &= M_4Q_4, & z'_2 &= N_4S_4, & z'_3 &= P_4R_4; \end{aligned}$$

we should have by the aid of auxiliary equations of the fifth degree, as above, the quantities w'_1, w'_2, w'_3; x'_1, x'_2, x'_3; y'_1, y'_2, y'_3; z'_1, z'_2, z'_3 all ultimately determined in terms of the coefficients of the original quintic equation.

Evidently by the solution of

$$\begin{aligned} \theta^2 - w_1\theta + w'_1 &= 0, \ M_1 \text{ becomes known;} \\ \theta^2 - x_1\theta + x'_1 &= 0, \ M_2 \quad \text{“} \quad \text{“} \\ \theta^2 - y_1\theta + y'_1 &= 0, \ M_3 \quad \text{“} \quad \text{“} \\ \theta^2 - z_1\theta + z'_1 &= 0, \ M_4 \quad \text{“} \quad \text{“} \end{aligned}$$

whence the four quartic roots of group M_1 are given by the quartic equation

$$\zeta^4 - M_1\zeta^3 + M_2\zeta^2 - M_3\zeta + M_4 = 0.$$

Thus without the aid of any intervening equations higher than the fifth degree we arrive at the final quartic equation, whose roots are evidently identically the *same* as the roots of the quartic employed by Lagrange after the resolution of his “Sextic.”

A THEOREM IN PLANE CUBICS.

By Prof. F. H. Loud, Colorado Springs, Colo.

Let a right line meet a cubic curve in the three points P, Q, R; from each of these points let a line be drawn, viz: From P a line meeting the curve in G and L, from Q one meeting it in H and M, from R one meeting it in K and N. Join these six points by twos as follows: Let the line KM meet the curve in A and meet the line GL in a; let GN meet the curve in B and meet HM in b; let HL meet the curve in C and meet KN in c. Then will (1) the points a, b, c lie on a right line, and (2) the points A, B, C lie on a right line, and (3) the lines abc, ABC, PQR meet in a point.

(1) In the following arrangement:

$$\begin{array}{cccc} P & Q & R \\ L & H & c \\ G & b & N \\ a & M & K, \end{array}$$

the letters placed in any horizontal row represent points which by the foregoing hypotheses lie in one right line; and so also do the letters which are in one vertical column. The columns therefore represent a cubic system and the rows a quartic system intersecting the former in twelve points. The given cubic curve with the line joining b and c forms another quartic system passing through eleven of these points. But a quartic passing through eleven fixed points on a cubic must pass through a twelfth (Salmon, H. P. C., p. 20, § 33); hence the last-named quartic system passes through a. But a cannot be a point of the cubic curve (which has already nine known points in common with the cubic system of lines), hence it lies on the line bc.

(2) Let the line LH meet the curve in C and the line GN meet it in B; also let the line BC meet the line MK in A. If it is proved that A is on the curve, the second part of the above theorem is demonstrated. In the following arrangement:

$$\begin{array}{cccc} L & H & c & C \\ G & b & N & B \\ a & M & K & A, \end{array}$$

the lines of one row or of one column represent collinear points as before, and we again have a cubic system intersected by a quartic; also the line abc and the cubic curve which passes through the points BC, GHK, LMN make up a quartic through eleven of these points. Hence this quartic passes through the twelfth point A, and as this cannot be a point of the line which intersects the cubic system in the three points a, b, c, it is a point of the cubic curve.

(3) Finally consider this last arrangement,

$$\begin{array}{cccc} P & Q & R & \\ L & H & c & C \\ G & b & N & B \\ a & M & K & A, \end{array}$$

in which as before the letters of one row or of one column represent collinear points. We have here two quartic systems of lines intersecting not only in the fifteen points of the diagram, but also in the point in which the line PQR meets ABC. The cubic curve and the line abc make up another quartic, passing through the fifteen points of the diagram, thirteen of which are independent. This last quartic system therefore passes through the sixteenth point above named, and as this cannot lie on the curve, which has twelve points of intersection with either quartic, it lies on the line. Hence the lines ABC, abc, PQR meet in a point.

It is to be observed that the proof of either the first or second part of the above theorem did not require the whole of the hypothesis as stated. Either of these parts might accordingly be stated as a separate theorem with a simpler hypothesis. The reciprocal theorem, concerning tangents to a curve of third class, and the subordinate theorems which may be obtained by placing at infinity some of the points or lines of the diagram, constitute a group of propositions of which a mere mention is sufficient.

COLORADO COLLEGE, *August* 1, 1890.

WEIERSTRASS'S ELLIPTIC INTEGRAL.

By Dr. Thomas S. Fiske, New York, N. Y.

It is proposed in the following note to apply the theory of the imaginary variable, so elaborately expounded by Briot and Bouquet,* to the definite integral by which Weierstrass's p-function is generally defined.† It is thought that the method of introducing the infinite values of the variable in connection with the formation of the elementary contours presents some novelty.

The definite integral in question is

$$u = \int_x^y \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} = \int_x^y \frac{2\,dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}},$$

Weierstrass's p-function being defined by the auxiliary equation

$$y = \mathrm{p}(u).$$

The initial point of the independent variable is regarded as situated upon the circumference of a circle of infinite radius, R, and having its centre at the origin. For any such point we may write

$$y = Re^{i\theta};$$

whence

$$\sqrt{(y-e_1)(y-e_2)(y-e_3)} = R^{\frac{3}{2}} e^{\frac{3}{2}i\theta}.$$

The value of the radical which results at any point of this circumference, by taking the argument of the variable positive and less than 2π, we may call its *positive* value. The radical also admits a value differing from the preceding in sign. This we will call the *negative* value.

The integral with respect to the entire circumference, or any part of it, is zero, for we have

$$u = \frac{i}{2R^{\frac{1}{2}}} \int e^{-\frac{1}{2}i\theta} d\theta,$$

* Théorie des Fonctions Elliptiques: Paris, 1875.

† Cf. Traité des Fonctions Elliptiques, G. H. Halphen, Vol. I, p. 55; Paris, 1886.

which, since R is infinite, is equal to zero. Hence the introduction of this circumference, or any part of it, at the beginning of the path of the variable, is equivalent simply to a change in the initial value of the radical. If two paths of the variable are identical except for the fact that one is preceded by this circumference, the corresponding values of the integral differ in algebraic sign.

The critical points of the differential function are e_1, e_2, e_3. The integral with respect to an infinitely small circumference described about any one of these points, or with respect to any part of such a circumference, is zero. For upon the circumference about one of these points, e_n, the independent variable takes the form

$$y = e_n + re^{i\theta},$$

in which r is infinitely small. Hence

$$\sqrt{(y-e_1)(y-e_2)(y-e_3)} = Er^{\frac{1}{2}}e^{\frac{1}{2}i\theta},$$

E denoting a quantity differing infinitely little from a finite constant, and

$$\int_{2_1} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}} = \frac{ir^{\frac{1}{2}}}{2E}\int e^{\frac{1}{2}i\theta}d\theta = 0.$$

The introduction of such a circumference, or any part of it, into the path of the variable, therefore simply changes the corresponding value of the radical.

Let us now consider the value of the integral when, the limit y having any value whatever, and the initial value of the radical being taken *positive*, the path of the independent variable is the exterior segment of the infinite radius drawn from the origin through y.* If we call the corresponding value of the integral $\mathrm{P}^{-1}(y)$, we have, substituting for the variable the product yt, in which t is a real quantity varying between infinity and unity,

$$\mathrm{P}^{-1}(y) = \frac{1}{2y^{\frac{1}{2}}}\int_{\infty}^{1} \frac{dt}{\sqrt{\left(t-\frac{e_1}{y}\right)\left(t-\frac{e_2}{y}\right)\left(t-\frac{e_3}{y}\right)}}.$$

If the rectilinear path along the radius includes any of the critical points e_1, e_2, e_3, the resulting indetermination is avoided by introducing into the path of the variable at each critical point an infinitely small semi-circumference

* Cf. Hermite, Cours à la Sorbonne, Quatrième édition, p. 76.

described about it in a positive direction as the variable moves toward the origin. To fix the significance of $\mathrm{P}^{-1}(0)$, that is when the limit y is at the origin, the path of the variable is taken along that radius which passes through real and positive quantities. Under these suppositions $\mathrm{P}^{-1}(y)$ has for every value of y a single determinate value. This value of the integral is not however obtainable solely by means of the path described above. Applying Cauchy's theorem upon definite integrals, we see that any two paths which lead from infinity to the same value of y, and which enclose no critical point between them, furnish the same value of the integral.

Let us now consider the value of the integral for an elementary contour, that is for a path of the variable which begins at the outer extremity of an infinite radius through one of the critical points and leads inward to a point infinitely near the critical point, around the circumference of an infinitely small circumference enclosing the critical point, and thereupon back to infinity along the path of approach. We thus obtain the three integrals,

$$2\mathrm{P}^{-1}(e_1) = e_1^{-\frac{1}{2}} \int_x^1 \frac{dt}{\sqrt{\left[t-1\right]\left[t-\frac{e_2}{e_1}\right]\left[t-\frac{e_3}{e_1}\right]}} = 2\omega_1,$$

$$2\mathrm{P}^{-1}(e_2) = e_2^{-\frac{1}{2}} \int_x^1 \frac{dt}{\sqrt{\left[t-\frac{e_1}{e_2}\right]\left[t-1\right]\left[t-\frac{e_3}{e_2}\right]}} = 2\omega_2,$$

$$2\mathrm{P}^{-1}(e_3) = e_3^{-\frac{1}{2}} \int_x^1 \frac{dt}{\sqrt{\left[t-\frac{e_1}{e_3}\right]\left[t-\frac{e_2}{e_3}\right]\left[t-1\right]}} = 2\omega_3.$$

By means of the theorem of Cauchy the most general path of the independent variable may be reduced to a combination of arcs of the infinite circumference and elementary contours, followed by a rectilinear path along the exterior segment of the infinite radius from the origin through the point y. The most general value of the integral for a given value of y consequently will be

$$2m_1\omega_1 + 2m_2\omega_2 + 2m_3\omega_3 \pm \mathrm{P}^{-1}(y),$$

where m_1, m_2, m_3 are any integers positive or negative. From this we see that the p-function is a periodic function satisfying the equation

$$\mathrm{p}(2m_1\omega_1 + 2m_2\omega_2 + 2m_3\omega_3 \pm u) = \mathrm{p}(u).$$

The periods $2\omega_1$, $2\omega_2$, $2\omega_3$ are not independent of one another. A path consisting of the infinite circumference described in the positive direction, in combination with all three of the elementary contours successively described in the negative direction, is equivalent to a closed path situated in the finite region of the plane and enclosing no critical points (Fig. 1). Hence the corresponding value of the integral is zero, and we have, when e_1, e_2, e_3 are in the order of increasing arguments, the relation

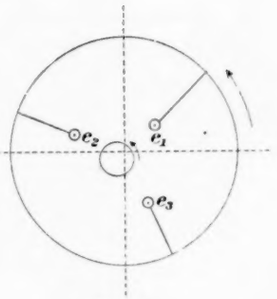

Fig. 1.

$$2\omega_1 - 2\omega_2 + 2\omega_3 = 0.$$

When the coefficients of the cubic

$$4y^3 - g_2 y - g_3$$

are real, the quantities e_1, e_2, e_3 may be all real, or one may be real and two conjugate imaginaries. In the first case the discriminant, $g_2^3 - 27g_3^2$, is positive; in the second case it is negative.

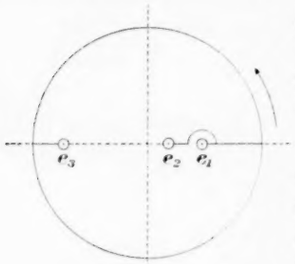

Fig. 2.

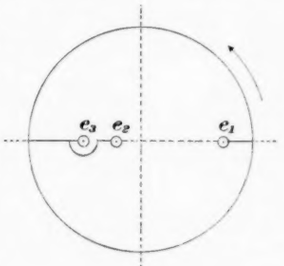

Fig. 3.

In the first case* the three quantities e_1, e_2, e_3 are all situated on the same diameter of the infinite circle, viz. the axis of reals. Let them be chosen so as to satisfy the condition $e_1 > e_2 > e_3$ (Figs. 2, 3). Of the two integrals,

$$2\omega_1 = \int_x^{e_1} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}}, \qquad 2\omega_3 = \int_x^{e_3} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}},$$

in each of which the independent variable is real, the former is real and the latter is a pure imaginary. From Fig. 2 we obtain, as above, $2\omega_1 - 2\omega_2 + 2\omega_3 = 0$, or $\omega_2 = \omega_1 + \omega_3$; in Fig. 3, however, this relation becomes $2\omega_1 - 2\omega_3 + 2\omega_2 = 0$, or $-\omega_2 = \omega_1 - \omega_3$.

* Cf. Halphen, Vol. I, pp. 55–57.

In the second case* let us suppose first that e_2 is real and positive (Fig. 4). ω_2 will be real. If now we take as an initial point negative real infinity, choosing the *positive* value of the radical, the quantity $2\omega_1 + 2\omega_3$ may be obtained not only from the path consisting of the infinite circumference described in a positive direction in connection with the elementary contours about e_1 and e_3, but also from the path consisting of the left-hand segment of the axis of reals taken twice but in opposite directions, and the infinitely small circumference enclosing e_2. The quantity $\omega_1 + \omega_3 = \omega'_2$, which is a semi-period independent of ω_2, is thus seen to be a pure imaginary. If, on the contrary, we suppose e_2 is negative (Fig. 5), we have ω_2 a pure imaginary, and $\omega_1 - \omega_3 = \omega'_2$, which results from the path formed by the right-hand segment of the axis of reals, a real quantity. We may write then under either supposition for the real and imaginary periods respectively,

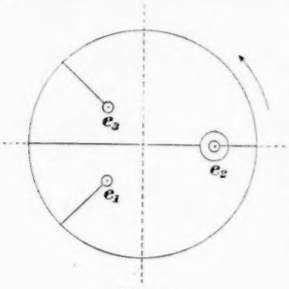

Fig. 4.

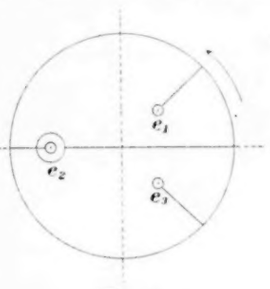

Fig. 5.

$$2\omega = \int_{x}^{e_2} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}}, \qquad 2\omega' = \int_{-x}^{e_2} \frac{dy}{\sqrt{(y-e_1)(y-e_2)(y-e_3)}},$$

in each of which the independent variable is real.

COLUMBIA COLLEGE.

*Cf. Halphen, Vol. I, pp. 69–72.

ON THE PARTIAL DERIVATIVES OF THE POTENTIAL FUNCTION IN THE PROBLEM OF n BODIES.

By Mr. W. J. Hussey, Ann Arbor, Mich.

Denoting the masses of the bodies by $m_1, m_2, \ldots m_n$, and their distances apart by $r_{12}, r_{13}, \ldots r_{(n-1)n}$, the potential function is

$$V = k^2 \left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \ldots + \frac{m_{n-1} m_n}{r_{(n-1)n}} \right),$$

in which k^2 is a constant.

Since V is a function of the distances, it is a function of the differences of the co-ordinates, and consequently the sum of its partial derivatives with respect to each co-ordinate is equal to zero. Therefore

$$\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} + \ldots + \frac{\partial V}{\partial x_n} = 0,$$

and similar expressions in y and z. These may be written

$$\left.\begin{aligned} \Sigma \frac{\partial V}{\partial x} &= 0, \\ \Sigma \frac{\partial V}{\partial y} &= 0, \\ \Sigma \frac{\partial V}{\partial z} &= 0. \end{aligned}\right\} \qquad (1)$$

Further, if the axes of co-ordinates are changed by a rotation about the axis of x, through an angle φ, y and z become

$$y = y' \cos \varphi - z' \sin \varphi,$$

$$z = y' \sin \varphi + z' \cos \varphi.$$

By this transformation V remains unchanged and independent of φ, consequently its partial derivative with respect to φ is equal to zero. This gives

$$\frac{\partial V}{\partial y_1}\frac{dy_1}{d\varphi} + \frac{\partial V}{\partial z_1}\frac{dz_1}{d\varphi} + \ldots = 0,$$

$$-z_1\frac{\partial V}{\partial y_1} + y_1\frac{\partial V}{\partial z_1} - \ldots = 0.$$

Or

$$\left.\begin{aligned} \Sigma\left(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y}\right) &= 0, \\ \Sigma\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right) &= 0, \\ \Sigma\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right) &= 0. \end{aligned}\right\} \qquad (2)$$

and similarly,

Equations (1) and (2) are of fundamental importance in the problem of n bodies, and this derivation of them is the most direct with which I am acquainted.

TWO NEW WORKS ON GRASSMANN'S GEOMETRICAL CALCULUS.*

Until very recently, Schlegel's *Raumlehre*† was the only available introduction to the study of Grassmann's works. Two new works of a similar character, though both quite independent of Schlegel and of each other, are therefore a welcome addition to the rapidly increasing Grassmann literature. For it is well known that few works in the whole realm of mathematical literature present as serious difficulties to even the more advanced student as do Grassmann's writings.

Prof. Hyde's Directional Calculus may be recommended to all students of the higher mathematics as admirably suited to serve as a first introduction to the study of Grassmann's methods as applied to geometry. It is arranged on a plan similar to that of the elementary text-books on Quaternions, such as Hardy's, or Kelland and Tait's. This method of treatment and the insertion of a large number of exercises make the book well adapted for use in the class-room. In fact, such use seems to have been the principal object of the author. For the purpose of private study, the complete lack of references to the literature of the subject, even to Grassmann's own writings, must be regarded as a serious defect which detracts somewhat from its usefulness.

In connection with this, another objection might be raised against the general method of treatment followed by Prof. Hyde. The study of Grassmann should not be undertaken before the student has attained at least some familiarity with modern higher mathematics. But then, it becomes of prime importance to bring out fully the relation of the *Ausdehnungslehre* to the methods of modern Geometry and Algebra. The occasional hints given by Prof. Hyde in this direction are far from sufficient.

And here exception might also be taken to the title adopted by Prof. Hyde. "Directional Calculus" would seem to be a name very remote from Grassmann's own conception of his geometrical methods. It is only suggestive of the road that led Prof. Hyde himself to the study of Grassmann. "The Author, though formerly an enthusiastic admirer of Hamilton's Quaternions, has been brought, by study and experience in teaching both, to a firm belief in the great practical, as well as theoretical, superiority of Grassmann's system" (*Directional Calculus*, Preface, p. IV). But the *Ausdehnungslehre*,

* PEANO, GIUSEPPE, Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva. Torino, Frat. Bocca, 1888. 8vo, pp. XII and 170.

HYDE, E. W., The Directional Calculus, based upon the methods of Hermann Grassmann. Boston, Ginn & Co., 1890. 8vo, pp. XII and 256.

† SCHLEGEL, VICTOR, System der Raumlehre. Leipzig, Teubner. 2 vols, 1872 and 1875.

even if restricted to its geometrical applications, is not merely a substitute for Quaternions. Hamilton's great creation was *a* method for studying geometrical questions; Grassmann's system gives us *the* geometrical calculus *par excellence*, of which all other methods are only special cases. Prof. Hyde himself seems to recognize this when he says (*l. c.*): "This superiority [of Grassmann's system] consists, according to the judgment of the writer, first, and largely, in the fact that Grassmann's system is founded upon, and absolutely consistent with, the idea of geometric dimensions. Second, in the fact that *all* geometric quantities appear as independent units, viz.: the point; the point at ∞, or line direction; the definite line; the line at ∞, or plane direction; the definite plane; and, finally, the plane at ∞, equivalent to a volume, which is a scalar. The same holds for space of any number of dimensions; in fact, it seems scarcely possible that any method can ever be devised, comparable with this, for investigating n-dimensional space."

Prof. Hyde's remarks on the comparative value of the methods of Hamilton and Grassmann are very just, and particularly valuable as coming from one who has made a thorough study of both.* And yet, he does not seem to have freed himself entirely of the idea of *co-ordinating* the two systems.

The view taken by Prof. Peano of his subject is much broader. The Italian author defines it as follows (Preface, p. IV): "The geometrical calculus consists of a system of operations to be performed on geometric objects, analogous to those applied to numbers in Algebra. It gives us the means of expressing the results of geometrical constructions by means of formulæ, of representing geometrical propositions by means of equations, and of substituting transformations of equations to ordinary reasoning. The geometrical calculus has some analogy to analytic geometry; but it differs from it in the fact that, while in analytic geometry all calculations are performed on certain numbers that determine the geometrical objects, in this new science the calculations are performed on the objects themselves."

Prof. Peano is perhaps rather too sanguine in his expectations as to the future of the "new science." But his words are worth quoting (ib., pp. VII, VIII): "In my opinion the time is not far when this geometrical calculus, or something analogous to it, will take the place of the methods actually in use in higher mathematical instruction. It is true, of course, that the study of this calculus, as that of any other science, requires a certain amount of time. But I do not think that more time is required for it than for the study of, say, the elements of analytic geometry. And it must be taken into account

* See his art. "Calculus of direction and position," in the Am. Journ. Math., vol. VI, p. 1.

that the student will then be in possession of a method which comprises that of analytic geometry as a special case, is far more powerful, and lends itself admirably to the study of the geometrical applications of infinitesimal analysis, of mechanics and graphical statics; in fact, some parts of these sciences would be already known to one that had studied this calculus."

Very likely, Prof. Hyde would be the first to subscribe to this view; and he is certainly to be congratulated upon his persistent efforts in introducing the "new science" to English-speaking mathematicians by repeated courses of lectures on the subject in the University of Cincinnati, and finally by a very useful and (in the best sense of the term) *practical* text-book.

The first chapter of the "Directional Calculus" gives a succinct account of geometrical addition. It must be remembered that the idea of geometrical addition of points, vectors, areas, etc., which in its various forms was suggested by the geometrical illustration of complex numbers and by the requirements of mechanics, was developed independently of Grassmann, and partly before the publication of the first edition of his *Ausdehnungslehre* (1844), by Möbius, Bellavitis, Hamilton, Resal, and others. It is therefore only in the second chapter, on "Multiplication," that we are introduced to the conceptions and methods most characteristic of Grassmann's work. This rather long chapter (pp. 23 to 77) forms the basis of all that follows; it is the most important in the book, but at the same time the most difficult for the author to write, and for the beginner to read. On the whole, the task is well accomplished by Prof. Hyde. The method he follows in introducing the new ideas is often one of suggestion more than of actual rigid proof, and this has the advantage of leading the student more rapidly to an actual working knowledge of the subject. Some teachers will probably prefer to select only parts of this chapter, dealing as it does with so many unfamiliar ideas, before taking up Chapter III, "Applications to Plane Geometry." In Chapter IV, "Scalar Point Equations," the conic section is very fully treated from a point of view different from that adopted in the preceding chapter, the treatment in Chap. III corresponding to that by means of ordinary Cartesian co-ordinates, while the "scalar point equations" give the equivalent to the use of trilinear co-ordinates.

It is to be regretted that here, as well as in the following two chapters on Solid Geometry, the author confines himself entirely to equations of the second degree. Grassmann's own investigations on higher plane curves,* and

* See his papers in Crelle's Journ., vols. 31 (1846), pp. 111–132; 36 (1848), pp. 177–182; 42 (1851), pp. 187–192; 52 (1856), pp. 254–275; and in *Gött. Nachr.*, 1872, p. 505.

his son's papers on twisted curves and the general theory of surfaces,* would have furnished interesting material for an additional chapter.

The last chapter, "Applications to Statics," though very well written, is exceedingly brief. It is worth mentioning, however, that the elements of R. S. Ball's theory of screws have found a place in Prof. Hyde's book; and the readiness and naturalness with which Grassmann's system lends itself to the treatment of this beautiful theory is a further proof, if any be needed, of its wide usefulness.

In selecting English terms for Grassmann's new conceptions the author seems, in general, to have been judicious, making use of some terms that have become familiar to English readers through the quaternion calculus, such as vector, scalar, tensor, etc., and coining new expressions wherever it could not be avoided. Unfortunately, Prof. Hyde's terminology is widely different from that of Prof. Peano, of which more will be said below.

In outward appearance Prof. Hyde's book is very pleasing, both paper and printing being a credit to the publishing house (Ginn & Co.). The proof is read carefully, and but few misprints have been noticed. But the insertion of eight or ten blank pages at the end of every chapter gives the book a rather odd appearance. And why deprive the tetrahedra of their h's? Why use the cumbersome symbol $\sin \angle^{\varepsilon_2}_{\varepsilon_1}$ where $\sin \varepsilon_1 \varepsilon_2$ would do just as well? Why say "bisectrix" for "bisector"? Such oddities do not add to the attractiveness of a book.

Prof. Peano's *Calcolo geometrico*, while pursuing the same object as the "Directional Calculus," is a work of a very different character. It is impossible in a brief review to give an adequate account of its varied and highly original contents.

The introduction, on the fundamental ideas and processes of Symbolic Logic, though it may deter some mathematicians from studying the main part of the work, is very interesting, in particular if read in connection with two little pamphlets by the author,† in which Boole's logical calculus is further developed and applied to the investigation of the principles of Algebra and Geometry. It is true that the philosophical form in which Grassmann himself presented his mathematical theories contributed in no small degree to

* Hermann Grassmann, [Jr.,] Raumkurven, 1886, and Krumme Flächen, 1888, Halle a. S. (School Programs).

† Peano, Ioseph, Arithmetices principia nova methodo exposita. Aug. Taur., Frat. Bocca, 1889. Pp. XVI and 20.

Peano, Giuseppe, I principii di geometria logicamente esposti. Torino, Frat. Bocca, 1889. Pp. 40.

make his first work unpopular among mathematicians and, for a time, to prevent its exerting the stimulating influence on research the author had justly expected (see the preface to the second edition (1878) of the *Ausdehnungslehre* of 1844, p. XV). But it must not be forgotten that by severing Grassmann's geometrical theories from their logical foundation, much of their real value is necessarily lost.

Since the time of the first appearance of the *Ausdehnungslehre* (1844) great progress has been made in extending the bounds of mathematical speculation and gaining a broader philosophic view of the science of mathematics. The thorough investigation of the axioms of geometry has gradually led to the idea of n-dimensional space and non-Euclidean geometry, while an n-dimensional algebra has grown out of the study of the imaginary. At the same time, the true character of the symbolic language of mathematical formulæ has come to be better understood, owing to the development of a "calculus of operations" with so-called "symbolic" methods in various branches of mathematics, such as quaternions, determinants, invariants, differential equations, theory of groups, etc. Mathematicians have thus become familiar with a conception and use of symbols that hardly differs from that of Symbolic Logic.* It may therefore be hoped that Prof. Peano's presentation of Boole's logical calculus may not be laid aside by the modern mathematician as something lying outside his sphere.

The plan adopted by Prof. Peano in presenting Grassmann's system of geometry may appear somewhat artificial at first glance. It has however great advantages for a rigorous and systematic development of the theory. The tetrahedral *volume* $ABCD$ is taken as the fundamental geometric object. It is defined as a scalar quantity by taking into account the sense in which it is generated. The ratio of two such scalars being a mere number, two volumes can be defined as being equal if their ratios to the same third volume are equal; and the sum of two volumes A and B is defined as that volume Γ for which the numerical equation $\frac{A}{\Omega} + \frac{B}{\Omega} = \frac{\Gamma}{\Omega}$ holds, where Ω is any given unit-volume. Similarly, the more general equation $\Sigma = mA + nB + p\Gamma + \dots$, where $m, n, p, \dots$ are numbers, can be interpreted as being equivalent to the numerical equation $\frac{\Sigma}{\Omega} = m\frac{A}{\Omega} + n\frac{B}{\Omega} + p\frac{\Gamma}{\Omega} + \dots$.

These definitions make it possible to apply to volumes certain operations

* See the interesting paper by Mr. A. B. Kempe, "Subject-Matter of Exact Thought," in *Nature*, vol. 43, No. 7 (Dec. 18, 1890), pp. 156–162.

that hold for numbers. For all other geometric objects, viz. *points*, "*lines*," and "*surfaces*," the rules of operation are derived by a very simple and elegant process from those originally established for volumes. The "*line*" is what Grassmann calls *Linientheil*, i. e. a localized vector (the "point-vector" of Prof. Hyde), while the "*surface*" is Grassmann's *Flächentheil*, or the "point-plane-vector" of Prof. Hyde.

Any expression of the form $mA + nB + pC + \ldots$ where $m, n, p, \ldots$ are numerical coefficients represents a *formazione geometrica*, let us say a *configuration;* it is said to be of the first, second, third, or fourth species, respectively, according as $A, B, C, \ldots$ are points, "lines," "surfaces," or volumes. The operations established for volumes are readily extended to configurations of all species; in addition, a new operation, Grassmann's "progressive" multiplication, is introduced. This forms the groundwork of the whole system.

The first three configurations are next taken up in separate chapters and fully discussed. It appears that a configuration of the first species in general represents a multiple point, but reduces to a *vector* (or point at infinity) whenever the sum of the coefficients of the configuration is $= 0$. In the configuration of the second species, the product of two vectors, or what amounts to the same, the sum of two equal and opposite "lines," plays a part analogous to that of the vector in the configuration of the first species. It is called a *bivector* by Prof. Peano ("plane-vector" by Prof. Hyde). It can be said that every configuration of the second species is in general reducible to a "line" and a "bivector," the latter being equivalent to a "line" at infinity. Similarly, in the configuration of the third species, the product of three vectors, or *trivector*, corresponds to a "surface" at infinity.

After the discussion of the configurations, a chapter each is devoted to the systems of the line, of the plane, and of space. Numerous "applications," or exercises, are appended to each chapter.

The last 45 pages of the book contain a chapter on the application of infinitesimal analysis to the geometrical configurations, and a chapter on linear transformations, with special reference to their application in geometry. This constitutes perhaps the most interesting, because the most original, part of Prof. Peano's work. A reference to the same author's larger treatise on the applications of the infinitesimal calculus to geometrical and mechanical investigations* must here suffice.

ALEXANDER ZIWET.

JAN. 31, 1891.

* PEANO, GIUS., Applicazioni geometriche del calcolo infinitesimale. Torino, Frat. Bocca, 1887. Pp. XII and 336.

BAKER'S ELLIPTIC FUNCTIONS.*

The appearance, which we here record, of the first work of considerable extent devoted exclusively to Elliptic Functions published in this country is a noteworthy event. The volume before us is stated in the preface to have been compiled "in the hope of smoothing the road to this interesting and increasingly important branch of Mathematics, and of putting within reach of the English student a tolerably complete outline of the subject, clothed in simple mathematical language and methods." The author adds: "New or original methods of treatment are not to be looked for. The most that can be expected will be the simplifying of methods and the reduction of them to such as will be intelligible to the average student of Higher Mathematics."

The design is excellent. It is to be regretted, however, that the carelessness of style displayed in the last sentence quoted above more or less pervades the whole volume. It would almost seem in some places as if the author had sent his rough mathematical notes to the printer, and omitted entirely to revise the proof-sheets. The most glaring instances of this we have noticed are these: On page 14, equation (17) is followed by a line of mathematical symbols which seem to have been dropped there by pure accident. On page 63 the author, having shown that $\frac{1}{A} = 1\ \overline{k}$, and desiring to show that $\frac{B}{A} = B1\overline{k}$, writes down the following intermediate step! "We have identically

$$1 = B\frac{1}{B} = B\frac{\frac{1}{A}}{\frac{B}{A}} = B1\frac{\overline{k}}{\frac{B}{A}};$$

whence

$$\frac{B}{A} = B1\ \overline{k}."$$

Aside from matters of this sort, the want of clearness of style is a serious drawback in a text-book. It by no means conduces to smoothing the road of the student, when he is allowed to puzzle himself over some statement so made as to lead him to suppose it ought to be self-evident, only to find, if he

* Elliptic Functions. An elementary text-book for students of mathematics, by Arthur L. Baker, C. E., Ph. D. 118 pages large 8vo. New York. John Wiley and Sons. 1890.

has courage to go on, that it is to form the conclusion of the argument into which he next plunges with no notion of its probable drift.

But it is more important to consider what should be the principle guiding the selection of matter in an elementary text-book upon a high mathematical subject. The decision of this point depends greatly upon the answer to a more fundamental question: How are we to regard the subject itself? Is it merely an attempt to surmount certain mathematical difficulties; to solve certain problems which may arise in the practical applications of mathematics? Or is it, on the other hand, a mathematical theory, worthy of study not only as an intellectual exercise, but for the sake of its own intrinsic beauty and symmetry?

In the lower branches we are inevitably compelled to take the former view to a great extent; the interest of the student must be aroused, if at all, by the solution of problems, the conquest of difficulties before unsurmountable. Weapons are being forged for him, he must be permitted to practise in their use, no matter how fantastical the enemies which are set up to be demolished by him.

This spirit is frequently carried too far, even in the preparation of treatises on the lower branches; and there is a tendency to introduce it into the higher subjects, where it is quite out of place, even in an elementary treatise. The effect is to overload the subject with matters which can be made to wear a practical look, thus sacrificing space which would have been used to far greater advantage, both in point of benefit and interest to the student, in giving at least an outline of the theory.

It seems to us Prof. Baker makes the mistake just indicated. Thus a great portion of his book is given up to the very dreary subject of the reduction of integrals to the standard elliptic forms, and a large space to Landen's Transformation and to numerical computations based upon the resulting formulæ. On the other hand, when the double periodicity is reached, the author says (having, however, stopped short of the equations which actually exhibit the imaginary period): "We see from what has preceded that Elliptic Functions have two periods, one a real period, and one an imaginary period. In the former characteristic they resemble Trigonometric Functions, and in the latter Logarithmic (? Exponential) Functions. On account of these two periods they are often called Doubly Periodic Functions. Some authors make this double periodicity the starting-point of their investigations. This method of investigation gives some very beautiful results and processes, but not of a kind adapted for an elementary work."

It is true that most mathematical theories have arisen from the attempt to surmount practical difficulties, and the consideration of these difficulties frequently suggests the most interesting method (largely because it is the historic method) of approaching the theory. For example, elliptic functions grew out of the attempt to perform integrations of an order transcending those which had already been accomplished. It does not follow that an elementary treatment of the subject should be subordinated to the consideration of these practical difficulties; nor, on the other hand, is it true that the theory which arose out of these attempts can only be properly developed by abandoning the historic point of departure. X.

SOLUTIONS OF EXERCISES.

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25

An elastic ring of radius a is placed gently on a smooth paraboloid of revolution whose axis is vertical; find, by use of the principle of energy, the lowest position to which the ring will descend, and its position of static equilibrium.

[*R. D. Bohannan.*]

SOLUTION.

The forces acting on the ring are gravity, its elastic tension and the normal resistance of the surface. In the descent of the ring from its initial position to its lowest position this latter force does no work, while the work done by gravity must equal the work done by the contractile force of the ring.

Let the contractile force of the ring be P and the ring-tension T, then

$$P = 2\pi T.$$

If x be the radius of the ring in its lowest position, then

$$T = \frac{\mu}{a}(x - a),$$

μ being the modulus of elasticity of the ring.

Equating gravity work to that of the ring-tension we have, if $x^2 = my$ be the meridian of the surface, and W the weight of the ring,

$$\frac{W}{m}(x^2 - a^2) = 2\pi \int_a^x \frac{\mu}{a}(x - a)\, dx.$$

Whence

$$x = \frac{1 + \dfrac{a\,W}{m\pi\mu}}{1 - \dfrac{a\,W}{m\pi\mu}}\, a.$$

Otherwise resolving gravity and P into the tangent cone of semi-angle α, we have for the force F urging the ring along the surface,

$$F = W\cos\alpha - P\sin\alpha.$$

The energy of the ring in its lowest position is equal to the work done in bringing it there, or

$$E = \tfrac{1}{2} Mv^2 = \int_a^x F ds = 0$$

$$= \int_a^x (W dy - P dx),$$

since $\cos \alpha = dy/ds$ and $\sin \alpha = dx/ds$, and the result is the same as above.

The condition for static equilibrium is

$$W \cos \alpha - P \sin \alpha = 0$$

or

$$W = P \tan \alpha.$$

$\tan \alpha = dx/dy$, from $x^2 = my$, is $m/2r$. Hence if x_0 be the radius of the ring in equilibrium

$$x_0 = \frac{a}{1 - \dfrac{a W}{m \pi \rho}}.$$

[*W. H. Echols.*]

131

Find the most general curve of the fourth degree which shall consist of two equal symmetrical loops and have no other branch, thus forming approximately a figure eight. Discuss the changes which result from causing the constants to vary.

SOLUTION.

The conditions of the problem make it easy to choose our axes so that the most general curve shall be represented by

$$Ax^4 + By^4 + Cx^2y^2 + Dx^2 + Ey^2 = 0,$$

in which, choosing B as unity, there remain four constants to consider. We introduce the dimensions in the figure as follows:—

First, if b be the length of the intercepts on the y axis, then will $E = -b^2$. Calling m the positive slope of the tangents at origin, $D = m^2b^2$.

The equation now becomes

$$Ax^4 + y^4 + Cx^2y^2 + b^2(m^2x^2 - y^2) = 0.$$

Next, introducing the condition that $x = a$ is a tangent to the curve at the points $(a, \rho a)$ and $(a, -\rho a)$, we have for these points $dx/dy = 0$; whence, $C = w^2 - 2\rho^2$, where $w = b/a$. Introducing the given values of the co-ordinates and coefficients, we get from the equation to the curve $A = \rho^4 - w^2m^2$.

We may now write

$$(\rho^4 - m^2w^2)\,x^4 + y^4 + (w^2 - 2\rho^2)\,x^2y^2 + b^2\,(m^2x^2 - y^2) = 0.$$

It is to be remarked that w, ρ, and m are in descending order of magnitude. To determine their limitations and relations, we observe that the curve must lie entirely within $y^2 = m^2x^2$ and $y^2 - b^2 = 0$. Now, in order that the curve may not intersect the axis of x except at the origin, $A = \rho^4 - m^2w^2$ must be positive, that is $\rho^2 > mw$. Again, that $y^2 - b^2 = 0$ may not cut the curve $\rho^2 < \frac{1}{2}(m^2 + w^2)$.

Putting $\rho^4 = m^2w^2 + c^4$, $2\rho^2 = m^2 + w^2 - d^2$, where c and d are real, and reducing, we have

$$(y^2 - b^2)\,(y^2 - m^2x^2) + x^2\,(c^4x^2 + d^2y^2) = 0,$$

a curve passing through the intersections of $x^2 = 0$ and the imaginary ellipse $c^4x^2 + d^2y^2 = 0$ with the four lines $(y^2 - b^2)\,(y^2 - m^2x^2) = 0$.

To best discuss the limitations to be put upon c and d, proceed as follows: Let $\rho^2 = mw + k^2$, where k is a third real magnitude. Combining this with $2\rho^2 = m^2 + w^2 - d^2$, we obtain

$$2k^2 + d^2 = (m - w)^2,$$

and

$$c^4 = k^2\,(k^2 + 2mw).$$

Suppose m given and $w = m + \Delta$, where Δ is a positive increment; then, finally,

$$2k^2 + d^2 = \Delta^2.$$

If Δ be assigned, it is easily seen how the curve changes. Since Δ may have all values from 0 to $+\infty$, so may c and d. But when $c = 0$, there are points at infinity. We remark the following cases:—

(a) When $\Delta = 0$, so are c and d, and the curve reduces to

$$(y^2 - b^2)\,(y^2 - m^2x^2) = 0.$$

(b) When Δ is infinite, the case $c^4 = \infty$ gives four lines $x^4 = 0$; and d^2 infinite gives in the same event $x^2y^2 = 0$.

It is interesting to notice the cases when $m = 0$. The two lobes are then tangent.

When $c = 1$, and $d = \pm 2$, we have two circles,

$$(x^2 + y^2)^2 = b^2y^2.$$

And in this case, if $c = 0$, we obtain the limiting curve

$$y^2\,(y^2 + d^2x^2 - b^2) = 0,$$

two straight lines and an ellipse. [*Percey F. Smith.*]

210

The length of a bar having the temperature t_o is l_o. Prove that when the bar rises to the temperature t, the length becomes

$$l = l_o e^{\varepsilon(t-t_o)},$$

e being the Napierian base of logarithms, and ε the linear coefficient of expansion, assumed constant. If ε is not constant, but a function of t, specify the conditions under which

$$l = l_o e^{[a(t-t_o)+b(t-t_o)^2+\ldots]}$$

and

$$l = l_o\,[1 + A\,(t-t_o) + B\,(t-t_o)^2 + \ldots]$$

hold true, a, b, $\ldots$ A, B, $\ldots$ being constants.

[*R. S. Woodward.*]

SOLUTION.

The conditions of the problem give

$$dl = l\varepsilon dt\,;$$

whence

$$\int_{l_o}^{l} \frac{dl}{l} = \varepsilon \int_{t_o}^{t} dt, \qquad \text{or} \qquad l = l_o e^{\varepsilon(t-t_o)}.$$

Now let $\varepsilon = f'(t)$. Then, as above, $l = l_o e^{f(t)-f(t_o)}$, and for the first case

$$f(t) - f(t_o) = a\,(t-t_o) + b\,(t-t_o)^2 + \ldots.$$

Differentiating, we find

$$f'(t) = \varepsilon = a + 2b\,(t-t_o) + 3c\,(t-t_o)^2 + \ldots.$$

For the second case,

$$f(t) - f(t_o) = \log_e [1 + A\,(t-t_o) + B\,(t-t_o)^2 + \ldots],$$

from which

$$f'(t) = \varepsilon = \frac{A + 2B\,(t-t_o) + 3C\,(t-t_o)^2 + \ldots}{1 + A\,(t-t_o) + B\,(t-t_o)^2 + \ldots}.$$

[*L. G. Weld.*]

246

Find the average distance of a given point in the surface of a circle from the circumference.

[*Artemas Martin.*]

SOLUTION.

Let P be the given point, O the centre of the circle. Let Q be any point in the circumference; join P and Q, and draw the radius OQ.

Put $OP = a$, $OQ = r$, and $\angle POQ = \varphi$.

Then $PQ = \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}$, and the average length of PQ is

$$J = \int_0^\pi \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}\, d\varphi \div \int_0^\pi d\varphi$$

$$= \frac{1}{\pi}\int_0^\pi \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}\, d\varphi.$$

Let $\varphi = \pi - 2\theta$, then $\cos\varphi = -\cos 2\theta = 2\sin^2\theta - 1$, and $d\varphi = -2d\theta$.

Substituting we have

$$J = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{(a^2 + r^2 + 2ar - 2ar\sin^2\theta)}\, d\theta$$

$$= \frac{2(a+r)}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{\left[1 - \frac{2ar}{(a+r)^2}\sin^2\theta\right]}\, d\theta$$

$$= \frac{2(a+r)}{\pi}\, \mathrm{E}\left[\sqrt{\frac{(2ar)}{a+r}}, \tfrac{1}{2}\pi\right].$$

[*Artemas Martin.*]

284

If the angular elevation of the summits of two spires (which appear in a straight line) is α, and the angular depressions of their reflections in a lake h feet below the point of observation are β and γ, then the horizontal distance between the spires is

$$2h\cos^2\alpha\sin(\beta - \gamma)\operatorname{cosec}(\beta - \alpha)\operatorname{cosec}(\gamma - \alpha).$$

SOLUTION.

Let B, C be the two spires, A the point of observation, A' the image of A in the lake; then in the triangle $BA'A$, $\angle B = \beta - \alpha$, $\angle A = \alpha + \frac{1}{2}\pi$; and in the triangle $CA'B$, $\angle C = \gamma - \alpha$, $\angle A' = \beta - \alpha$; and $AA' = 2h$. By the proportionality of the sides of any triangle to the sines of the opposite angles,

$$A'B = 2h\frac{\cos\alpha}{\sin(\beta - \alpha)}, \qquad BC = A'B\frac{\sin(\beta - \gamma)}{\sin(\gamma - \alpha)}, \qquad BD = BC\cos\alpha.$$

Hence the required distance BD is

$$2h\cos^2\alpha\sin(\beta - \gamma)\operatorname{cosec}(\beta - \alpha)\operatorname{cosec}(\gamma - \alpha).$$

[*R. B. Goodman.*]

285

Let a, b, c be the sides of a plane triangle, and α, β, γ the lines bisecting the angles and terminating in the opposite sides, then will

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)}{(a+b)(b+c)(c+a)} \times \text{area of triangle.}$$

SOLUTION.

Let the bisector from A cut BC in D. Then

$$AD = CD \sin C \operatorname{cosec} \tfrac{1}{2}A ;$$

$$\therefore \quad a = \frac{ab}{b+c} \sin C \operatorname{cosec} \tfrac{1}{2}A.$$

Taking similar expressions for β and γ and multiplying, we have

$$\alpha\beta\gamma = \frac{a^2b^2c^2}{(a+b)(b+c)(c+a)} \sin A \sin B \sin C \operatorname{cosec} \tfrac{1}{2}A \operatorname{cosec} \tfrac{1}{2}B \operatorname{cosec} \tfrac{1}{2}C.$$

Substituting the values of $\operatorname{cosec} \frac{1}{2}A$, etc., and remembering that

$$K = \text{area of triangle} = 2R^2 \sin A \sin B \sin C,$$

we have, after reduction,

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)K}{(a+b)(b+c)(c+a)}.$$

[*T. U. Taylor.*]

306

A body at distance r from the sun is moving with velocity v. Prove that the major axis of the orbit described is parallel to the direction of motion if, and only if, the velocity is "circular velocity for the distance r."

[*Ellery W. Davis.*]

SOLUTION.

The square of the velocity in the elliptical orbit, semi-major axis a, strength of centre μ, radius vector r, is

$$v^2 = \frac{\mu(2a - r)}{ar}. \qquad (1)$$

When the body moves parallel to the major axis, it is at one of the extremities of the minor axis, where $r = a$, and then, and only then, (1) gives $v = \sqrt{\mu/a}$, the velocity in a circle, radius a, about a force in the centre equal to μ/a^2.

[*William Hoover.*]

307

If a horizontal beam of length $2a$ is supported at each end, and has a load in the form of an isosceles triangle, base $2a$, height b, a unit's thickness throughout, and heaviness unity; show that the deflection of the beam due to this triangular load is $\frac{2a^4b}{15EI}$.

[*T. U. Taylor.*]

SOLUTION.

Take the origin at one of the supports, and let the axis of x be horizontal, that of y vertical. The moment at any point $(x, 0)$ will be

$$\frac{abx}{2} - \frac{bx^2}{2a} \cdot \frac{x}{3} = EI\frac{d^2y}{dx^2}.$$

Whence, by integration,

$$EI\frac{dy}{dx} = \frac{ab}{2} \cdot \frac{x^2}{2} - \frac{b}{6a} \cdot \frac{x^4}{4} + C. \tag{1}$$

If we put $x = a$, which makes $\frac{dy}{dx} = 0$, we have

$$C = -\frac{5a^3b}{24}.$$

Replacing C in (1) and integrating, since $y = 0$ for $x = 0$, we have

$$EIy = \frac{ab}{2} \cdot \frac{x^3}{6} - \frac{b}{6a} \cdot \frac{x^5}{20} - \frac{5a^3b}{24}x. \tag{2}$$

The equation (2) gives the ordinate of deflection for any value of the abscissa. At the middle we have $x = a$;

$$\therefore\ y = -\frac{2a^4b}{15EI},$$

the minus sign showing that the beam is bent downward.

[*C. L. De Mott.*]

308

Let points represent complex quantities in the usual way. Show that the quartic whose zeros are any four cotangential points on a fixed circular cubic, has a fixed Jacobian. [*F. Morley.*]

SOLUTION.

Let a, b, c, d be the points. Let $\overline{bc}, \overline{ad}$ meet at l; $\overline{ca}, \overline{bd}$ at m; $\overline{ab}, \overline{cd}$ at n. Let $p = (bc - ad)/(b + c - a - d)$, etc. Then we know (see Russell, London Math. Soc. xix) that the circles c, a, n; b, d, n; a, b, m; c, d, m meet at the point p.

Consider now a circular cubic of which a, b, c, d are cotangential points. Then l, m, n are also on the cubic. Since the circle c, a, n and the line b, d, m make up one circular cubic, and the circle b, d, n and the line a, b, m make up another, the point p is on the given cubic, and is the residual of a, b, c, d, m, n and the circular points at infinity.

Let o be the point on the cubic residual with m, n; and let ∞ be the real point at infinity. Then (see Salmon, Curves, Third Edition, p. 139) we may determine p as follows: Let l' be the residual of l, l; k that of o, ∞. Then p is the residual of l', k.

Now o and l' are cotangential (Salmon, p. 132). Hence, joining them to k, ∞ and p are cotangential.

Thus the centres of the 3 pairs of Jacobian points are the points cotangential with the real point at infinity. These are marked S, U, V in the figure in Salmon, p. 249. The Jacobian points are the geometric means of each pair of these points with regard to the third. See Russell, p. 60.

[*F. Morley.*]

309

A HORIZONTAL beam of length $2a$, supported at each end, has a load in the form of an inverted parabola symmetrical with respect to the vertical line through the centre of beam. If the vertex of the parabola is b above beam, and if the load is a unit's thickness and has a heaviness unity, the deflection of the beam due to the parabolic load is $\dfrac{61a^4b}{360EI}$. [*T. U. Taylor.*]

SOLUTION.

The equations for solution are (with the usual notation)

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = w, \qquad EI\frac{d^2\delta}{dx^2} = M;$$

in which $w = b - px^2$, if $px^2 = y$ be the equation to the parabola referred to tangent at vertex and vertical there as axes.

$$\begin{aligned} M &= \int\int w\,dx\,dx \\ &= \int\int (b - px^2)\,dx\,dx \\ &= \int (bx - \tfrac{1}{3}px^3)\,dx, \end{aligned}$$

the constant being zero since F and x are zero together;

$$\therefore M = \tfrac{1}{2}bx^2 - \tfrac{1}{12}px^4 - \tfrac{1}{2}ba^2 + \tfrac{1}{12}pa^4,$$

the constant being determined by putting $M = 0$, $x = a$.

$$\begin{aligned} EI\delta &= \int\int M\,dx\,dx \\ &= \int (\tfrac{1}{6}bx^3 - \tfrac{1}{60}px^5 - \tfrac{1}{2}ba^2x + \tfrac{1}{12}pa^4x)\,dx, \end{aligned}$$

the constant being zero since $d\delta/dx = 0$ for $x = 0$;

$$\therefore EI\delta = \tfrac{1}{24}bx^4 - \tfrac{1}{360}px^6 - \tfrac{1}{4}ba^2x^2 + \tfrac{1}{24}pa^4x^2 + C.$$

$x = 0$ gives δ for mid-span

$$EI\delta = C,$$

and since $\delta = 0$ for $x = a$ we have at mid-span

$$EI\delta = -\tfrac{1}{24}ba^4 + \tfrac{1}{360}pa^6 + \tfrac{1}{4}ba^4 - \tfrac{1}{24}pa^6.$$

If the parabola passes through the ends of the beam, as it is evidently intended that it should in the exercise, we have $p = b/a^2$. Whence, as required, $EI\delta = \tfrac{61}{360}a^4b$. [*W. H. Echols.*]

EXERCISES.

310

SOLVE for $x_1 : x_2 : x_3$ the continued equality

$$(x_2 + x_3)(x_2 - x_3)^2 = (x_3 + x_1)(x_3 - x_1)^2 = (x_1 + x_2)(x_1 - x_2)^2.$$

[*E. H. Moore.*]

311

TO FIND four biquadrate numbers whose sum is a square number.

[*Artemas Martin.*]

312*

GIVEN

$$u = x \sin x + \cos x,$$
$$v = \sin x - x \cos x;$$

to compute

$$\int \frac{x^2 dx}{u^2}, \qquad \int \frac{x^2 dx}{v^2}, \qquad \int \frac{bx^2 dx}{(au + bv)^2}.$$ [*C. Hermite.*]

313

GIVEN three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter. [*W. W. Johnson.*]

314

A CYLINDER, diameter $2b$, intersects a sphere, diameter $2a$, the surface of the cylinder passing through the centre of the sphere. Required the part of the volume of the sphere contained by the cylinder. [*Artemas Martin.*]

* Suggested by *W. W. Beman.* Hermite (Cours d'Analyse, 1873, p. 260) says: "On n'a aucun procédé pour trouver directement."

315

A MAN runs a race starting with velocity v, the direction of the wind appearing to him to make an angle a with his course. It then appears uniformly to veer round through an angle a during the race. The wind blows uniformly throughout and at right angles to his course. Show that if t be the time of the race, the length of the course is $vt \frac{\tan a}{a} \log (2 \cos a)$.

[*William Hoover.*]

316*

THE distance between the heels of a man's feet is $2b$, and the length of each foot is a. As the body sways, the vertical through the centre of gravity must always pass through the area contained by the feet. The toes should therefore be turned out at such an angle that the area contained by the feet is a maximum. Show (1) that a circle can be described about the feet with its centre on the straight line joining the toes, and (2) that the diameter of the circle is $b + (2a^2 + b^2)^{\frac{1}{2}}$. [*E. J. Routh.*]

317

THE arc of a lemniscate is shown in works on the Calculus to be equivalent to the arc of a certain ellipse. Show that the double point on the lemniscate corresponds with Fagnani's point on the ellipse.

[*W. B. Richards.*]

318

D is any point in the base BC of a triangle ABC. O_1 and O_2 are the centres of the inscribed circles of BAD and CAD. Then, if $\triangle$ = area ABC,

(1) Area $AO_1O_2 = \frac{\triangle r}{a} \operatorname{cosec} ADC$;

(2) If AD bisects A, $AO_1O_2 = \frac{\triangle r}{b+c} \operatorname{cosec} \frac{1}{2}A$;

(3) If AD is the altitude, $AO_1O_2 = \frac{\triangle r}{a}$.

[*T. U. Taylor.*]

319

To construct a triangle ABC ($BC > AB > CA$), the angle A being known, and having given

$AB + BC = m$, $\quad AB + CA = n$;

or $\quad BC - AB = p$, $\quad AB - CA = q$.

[*J. E. Hendricks.*]

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*Suggested by *A. Hall.*

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